Enhancing synchronizability by weight randomization on regular networks

Article *in* Physics of Condensed Matter · January 2007

DOI: 10.1140/epjb/e2007-00189-0

5 authors, including:

Li Daqing
Beihang University (BUAA)
70 PUBLICATIONS **1,018** CITATIONS

Menghui Li
Beijing Institute of Science and Technology Intelligence, Beijing, China
46 PUBLICATIONS **751** CITATIONS

Jinshan Wu
Beijing Normal University
59 PUBLICATIONS **621** CITATIONS

Zengru Di
Beijing Normal University
164 PUBLICATIONS **1,961** CITATIONS

Eur. Phys. J. B **57**, 423–428 (2007)
DOI: 10.1140/epjb/e2007-00189-0

THE EUROPEAN
PHYSICAL JOURNAL B

Enhancing synchronizability by weight randomization on regular networks

D.Q. Li[1], M.H. Li[1], J.S. Wu[2], Z.R. Di[1], and Y. Fan[1,a]

[1] Department of Systems Science, School of Management, Beijing Normal University, Beijing 100875, P.R. China
[2] Department of Physics and Astronomy, University of British Columbia, Vancouver, B.C. Canada, V6T 1Z1, Canada

Received 22 October 2006 / Received in final form 29 January 2007
Published online 13 July 2007 –

Abstract. In weighted networks, redistribution of link weights can effectively change the properties of networks, even though the corresponding binary topology remains unchanged. In this paper, the effects of weight randomization on synchronization of coupled chaotic maps is investigated on regular weighted networks. The results reveal that synchronizability is enhanced by redistributing of link weights, i.e. coupled maps reach complete synchronization with lower cost. Furthermore, we show numerically that the heterogeneity of link weights could improve the complete synchronization on regular weighted networks.

PACS. 89.75.Hc Networks and genealogical trees – 05.45.-a Nonlinear dynamics and chaos – 05.45.Xt Synchronization; coupled oscillators

1 Introduction

In recent decades, the significance of the concept of collective and self-organized behavior such as synchronization [1] has been realized in many different subjects. Recently, as the explosion of studies of complex networks [2–5], the part of synchronization of complex networks has provided a fresh framework for research on collective phenomena arising in many fields, such as biology, chemistry and sociology. Previous works about the synchronization of complex networks focused on the influence of topological connections on synchronization. Compared to regular networks, the ability to synchronize is generally enhanced in both small-world networks (SWNs) [6,7] and scale-free networks (SFNs) [5]. These results imply that synchronizability strongly depends on the average path length between oscillators. Further study indicates that the heterogenerous degree distribution decreases synchronizability [8]. Besides the continuous chaotic dynamics mentioned above, the synchronization of coupled chaotic maps has been fully studied in a variety of networks, including regular networks, scale-free networks, small-world networks, tree networks, and random networks [9].

However, synchronization, as many other dynamic processes in networks, is influenced not only by the topology, but also by the link weights of the network. In some studies, synchronizability is enhanced in different kinds of networks by choosing link weights based on the knowledge of the network topology [10–12]. These investigations are significant because most complex networks, which are relevant to synchronization, are indeed weighted, and link weights display highly heterogeneous, such as brain networks that allows coherent oscillation of excitable neurons [13], airport networks that underlie the synchronization of epidemic outbreaks in different cities [14] and technological networks whose functioning rely on the synchronization of interacting units [15]. Previous works have attempted some methods to investigate the influence of link weights on synchronization. Atternatively, redistributing of link weights provides another way to adjust properties of weighted network [18]. In this paper, we study the effect of redistributing link weights on network synchronizability, and find it increasing on regular networks with randomizing link weights.

Some fundamental problems in the context of complete synchronization of identical oscillators are discussed in references [16,17]. In this paper, a similar approach is used to study the synchronization of Logistic maps. Redistribution of link weights can produce a “Small-World” effect, similar to that of rewiring links on a regular network [18]. For chaotic oscillators on a weighted lattice, the synchronizability of systems is reinforced after redistributing the link weights. Randomization of link weights could reduce the critical coupling strength for complete synchronization, allowing the system to reach complete synchronization at lower cost. Using a uniform random distribution of link weights, we also provide an illustration that synchronizability could be enhanced by increasing the heterogeneity of link weights.

The paper is organized as follows. In Section 2, we introduce the method of redistributing link weights of

[a] e-mail: yfan@bnu.edu.cn

networks and show the Small-World effect produced by this procedure. The method of evaluating the ability to completely synchronize in weighted networks is formulated and analyzed in Section 3. In Section 4, we demonstrate that the synchronizability is enhanced with redistributing link weights. The heterogeneity of weight could make systems more synchronizable. The conclusions are presented in the last section.

2 Redistribution of link weights

Similar to the construction of Watts-Strogatz (WS) small-world networks [4], we redistribute link weights instead of rewiring links. The initial setup is a ring lattice with N vertices, k edges per vertex, each edge having the same dissimilarity weight w [18]. Here, we assume there is a minimum unit of weight, Δw. Starting from this original network, the procedure of redistributing link weights is as follows:

1. every unit of weight Δw in the original lattice is removed from the original link with probability P and transferred to a link randomly chosen over the whole lattice;
2. step 1 is repeated until each Δw in the original lattice has been tried once, except for the reallocated units of weight;
3. if the unit of weight Δw is the only unit of weight left on that link, it will not be attempted. This could avoid disconnecting a link in order to ensure the same corresponding binary network.

Through above procedure, without changing the binary structure, a regular network is adjusted to a new state with random weight distribution ($P = 1$) from that with δ weight distribution ($P = 0$). Through investigation of the intermediate region $0 < P < 1$, we can determine the effect of weight redistribution.

The link weight distribution after applying the above procedure can be determined by the following arguments. $T = Nk/2$ is the total number of links in the network. Let W_r denote the total number of units of minimum weight Δw removed from the original link, thus $W_r = wTP/\Delta w$. After redistribution of link weights, each link has weight $w_i = s_i \Delta w$, where s_i is the number of Δw on link i. The average of s_i is equal to $w/\Delta w$. Obviously, given Δw, the distribution of link weights is determined by the distribution of s. In this case, s can be divided into two parts: s_1 is the number of Δw left on links when the others are removed from this link; s_2 is the number of Δw transferred to other links. Thus $s = s_1 + s_2$. The probability distributions of s_1 and s_2 are respectively:

$$P(s_1) = C^{s_1}_{w/\Delta w}(1-P)^{s_1} P^{w/\Delta w - s_1}, \quad (1)$$

$$P(s_2) = C^{s_2}_{W_r}(1/T)^{s_2}(1-1/T)^{W_r - s_2} = \frac{e^{-\lambda}\lambda^{s_2}}{s_2!}, \quad (2)$$

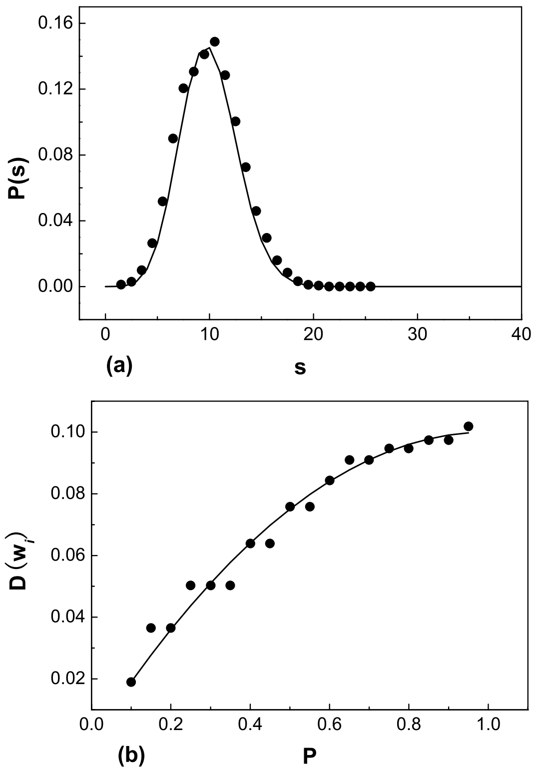

Fig. 1. (a) The distribution of s. (b) The variance of link weights. The line s shows the theoretical result given by equations (3) and (4), while the dots are the results of numerical simulation. ($N = 300, k = 120, w = 10, \Delta w = 1$. The dots are averaged over 20 realizations of randomization processes.)

where $\lambda = \frac{W_r}{T} = \frac{pw}{\Delta w}$ for large N. Combining these two parts, the distribution of s is as follows:

$$P(s) = \sum_{n=0}^{f(\Delta w, s)} C^n_{w/\Delta w}(1-P)^n P^{(w/\Delta w)-n} \frac{e^{-\lambda}\lambda^{(s-n)}}{(s-n)!}, \quad (3)$$

where $f(\Delta w, s) = min\{w/\Delta w, s\}$.

For $w_i = \Delta w s_i$, the distribution of weight randomized is the same as the distribution of s, which is the number of Δw that each link has finally. The shape of s distribution is similar to the degree distribution of the WS Small-World Model. It has a pronounced peak at $\langle s \rangle = w/\Delta w$ and decays exponentially for large s in Figure 1a. The variance of link weights w_i after redistribution is

$$D(w_i) = w \Delta w p(2 - P). \quad (4)$$

From Figure 1b, we can conclude that the heterogeneity of weight increases with P.

The above procedure of weight randomization could obviously change the structural properties of networks, such as the average weighted path length and the weighted clustering coefficient. After redistributing the dissimilarity weight over the whole network, the weighted distance of a path can easily be calculated from the sum of dissimilarity weight for any given path. The weighted clustering

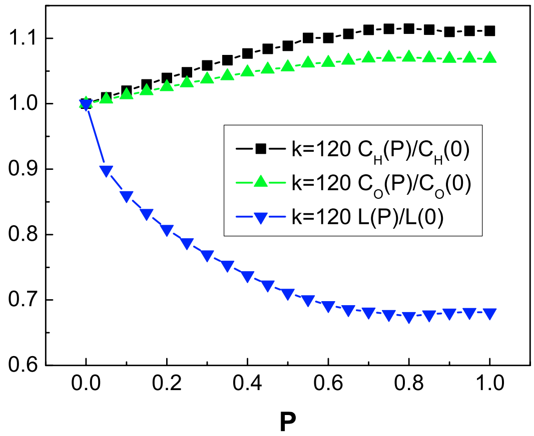

Fig. 2. The change of average weighted shortest path length $L(P)/L(0)$ and average weighted clustering coefficient $C(P)/C(0)$ as a function of P. The results are averaged over 20 random realizations of the redistribution processes ($N = 300, w = 10, \Delta w = 1$).

coefficient is calculated according to formulae initially proposed by Holme [19] and Onnela [20], and generalized as follows [18]:

$$C_H^w(i) = \frac{\sum_{j,k} \tilde{w}_{ij}\tilde{w}_{jk}\tilde{w}_{ki}}{\sum_{j,k} \tilde{w}_{ij}\tilde{w}_{ki}} \quad (5)$$

and

$$C_O^w(i) = \frac{2}{k_i(k_i-1)} \sum_{j,k} (\tilde{w}_{ij}\tilde{w}_{jk}\tilde{w}_{ki})^{1/3}, \quad (6)$$

where $\tilde{w}_{ij} = \frac{1}{w_{ij}}$ denotes similarity weight. Similar to the WS small-world network, the average weighted path length $L(P)$ and the weighted clustering coefficient $C(P)$ are used to present the structural properties of the network. Figure 2 shows that with redistributing link weights, the average path length is obviously decreased, while the clustering coefficient is increased. It demonstrates that besides random rewiring of links, randomizing weight will also lead to small-world phenomenon [18].

3 Synchronizability of chaotic maps in the network

In this section, we introduce a weighted model of coupled chaotic maps in regular networks. We then briefly present an evaluation for linear stability of completely synchronized states in terms of the eigenvalues of the coupling matrix and the Lyapunov exponent of the chaotic map.

The dynamics of a vertex in weighted network is described via:

$$x_i(t+1) = f(x_i(t)) + \frac{\gamma}{m} \sum_{j \neq i}^{m} J_{ij}(f(x_j(t)) - f(x_i(t))) \quad (7)$$

where $x_i(t)$ is a state variable and t denotes the discrete time. $f(x)$ describes the local dynamics, and is chosen using the logistic map $f(x) = \alpha x(1-x)$ with $\alpha = 3.9$. γ is the overall coupling strength. Here we take the coupling of the nearest and next nearest neighbors, so m is the number of nearest-neighbors and next nearest-neighbors of any vertex. Given the dissimilarity weight w_{ij} between any two vertices i and j connected directly, the interaction strength is inversely proportional to the distance between any two vertices. We use the link-preferential interaction as follows: J_{ij} is taken as $1/w_{ij}$ if i and j are connected directly (even though they may also be connected through a third vertex s), is taken as $1/min(w_{is}+w_{sj})$ if i and j are next nearest-neighbors and is set to 0 for other cases [18].

Next, we present the stability analysis of synchronized states in regular networks. Within a matrix formalism, the formula (7) can be written as:

$$X(t+1) = \mathbf{B}F(X(t)) \quad (8)$$

where $B_{ij} = [\delta_{ij}(1 - \frac{\gamma}{m}\sum_{j\neq i}^{m} J_{ij}) + \frac{\gamma J_{ij}}{m}]$ with $X(t+1) = [x_1(t+1), x_2(t+1), \ldots, x_n(t+1)]^T$ and $F(X(t)) = [f(x_1(t)), f(x_2(t)), \ldots, f(x_n(t))]^T$. Here B is the coupling matrix combining both topology and weight. The rows of B have the same sum, which ensures that the completely synchronized state $\{x_1 = x_2 = \ldots = x_n\}$ is an invariant manifold of Equation (8). The only eigenmode that corresponds to the uniform state is $[1, 1, \ldots, 1]$. The Jacobian matrix of the synchronized state J_t is directly related to the coupling matrix B by $J_t = BF'(X(t))$.

Let λ_0 be the eigenvalue corresponding to the eigenmode $[1, 1, \ldots, 1]$ and let $\lambda_i (i = 1, 2, \ldots, N-1)$ represent the other $N-1$ eigenvalues of the coupling matrix, ordered such that $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_{N-1}|$. Let λ represent the Lyapunov exponent of the map f. The stability of this state depends on the eigenvalues of coupling matrix and Lyapunov exponent λ. In terms of eigenvalues of the coupling matrix, Lyapunov exponents λ can be written as:

$$\begin{aligned} LE_i &= \lim_{t\to\infty} \tfrac{1}{t} \ln \prod_t |\lambda_i f'(x_i(t))| \\ &= \ln |\lambda_i e^{\lambda}|. \end{aligned} \quad (9)$$

The necessary condition for the stability of synchronous chaos is that only one eigenvalue fulfils $|\lambda_0 e^{\lambda}| > 1$ and other eigenvalues fulfil $|\lambda_i e^{\lambda}| < 1$ $(i = 1, 2, \ldots, N-1)$. In fact, $(ln|\lambda_1| + \lambda) < 0$ ensures $|\lambda_i e^{\lambda}| < 1$ $(i = 1, 2, \ldots, N-1)$. So the smaller $|\lambda_1|$ is, the more synchronizable the system is, and vice versa. As a result, we can measure the synchronizability of the system by calculating eigenvalues of the coupling matrix B.

Besides the stability of synchronization given by the eigenvalue, the efficiency of synchronization is also of great importance. To assess efficiency for synchronization of oscillators on weighted networks, we examine the minimal cost C needed to achieve complete synchronization [10]. The cost C is defined as $C = \gamma_{min} \sum_i \sum_j J_{ij}$, where γ_{min} is the minimum overall coupling strength that the synchronization requires. It can be used as a complementary measure of synchronizability. Obviously, it is more efficient when the oscillators are synchronized with lower cost C.

In the next section, we will apply these two approaches to study the synchronization ability of regular networks before and after redistributing link weights.

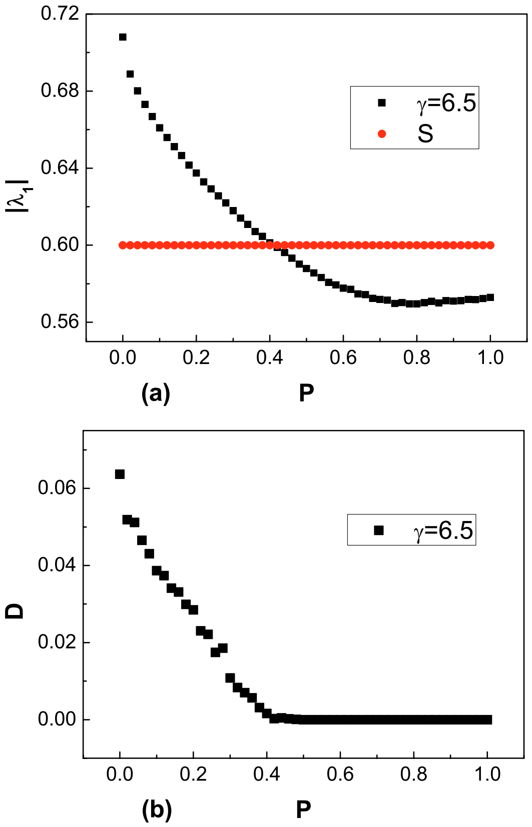

Fig. 3. (a) Eigenvalue $|\lambda_1|$ as a function of probability of randomization with coupling strength $\gamma = 6.5$. Curve S is the threshold for the stability of the synchronized state. ($N = 300, k = 120$). (b) Degre of synchronization as a function of probability of randomization with the same parameters. The threshold for stability is the same probability as the threshold for $D = 0$. Removing the relaxation time $t < 500$, we calculate D when $500 < t < 2000$ for approximation.

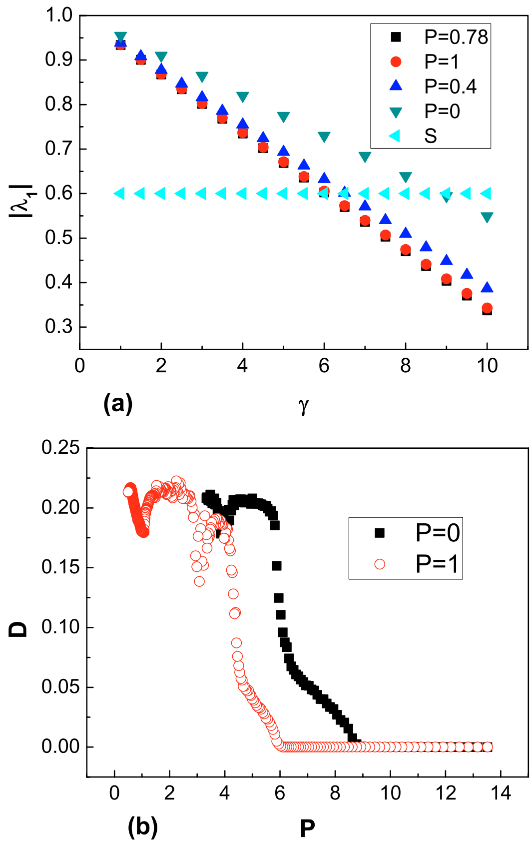

Fig. 4. (a) Eigenvalue $|\lambda_1|$ as a function of coupling strength γ with $p = 0$ and $p = 1$ respectively. (b) Degree of synchronization as a function of coupling strength γ with $p = 0$ and $p = 1$.

4 Effect of weight redistribution on synchronizability

As mentioned in the introduction, the topology of small-world networks can enhance their ability to synchronize. Here we investigate the "Small-World" effect induced by weight randomization on synchronization. We find that redistributing link weights could enhance synchronizability of chaotic maps on regular networks.

We start from a ring lattice with $N = 300$ vertices and $k = 120$ edges per vertex, where each link has the same dissimilarity weight $w = 10$ in the initial network. The results are averaged over 50 random redistribution processes. We then carry out the randomizing procedure described in Section 2 and study its effect on the dynamical process of synchronization. Since the eigenvalue $|\lambda_1|$ of the coupling matrix indicate the stability of synchronization, we now study the effect of redistributing link weights on $|\lambda_1|$. In Figure 3a, we find that redistribution of link weights has continuous effect on $|\lambda_1|$. It can be clearly observed that the stability of synchronous states is reinforced by randomizing link weights. Our numerical computations show that $|\lambda_1|$ decreases with increasing P and approaches as a minimum at around $P = 0.78$, similar to the behavior of the average path length as shown in Figure 2. This implies that synchronizability increases with decreasing average path length. In addition, we investigated the effect of overall coupling strength γ on $|\lambda_1|$ at different weight distributions. Figure 4a indicates that $|\lambda_1|$ decreases at almost every value of γ on account of randomization of link weights. The slope of the curve for $P = 1$ is steeper than that of the curve for $P = 0$. In addition, when γ is larger than 6, the curve for $P = 0.78$ is under the curve for $P = 1$, indicating again that $|\lambda_1|$ has a minimum at around $P = 0.78$. This implies that the more heterogeneous the link weights distribution is, the lower γ the synchronization state of the oscillators on the network requires. The reason may be that redistributing link weights can decrease the average path length. Therefore, the network gets "smaller" and more synchronizable.

Furthermore, the cost C is reduced when the link weights is randomized (as shown in Fig. 5). This is interesting because the minimal cost and the maximum synchronizability occur correspondingly with increasing P. We not only enhance the synchronizability but also decrease the cost, which is one of the most important ingredients in realistic networks. In addition, since the speed

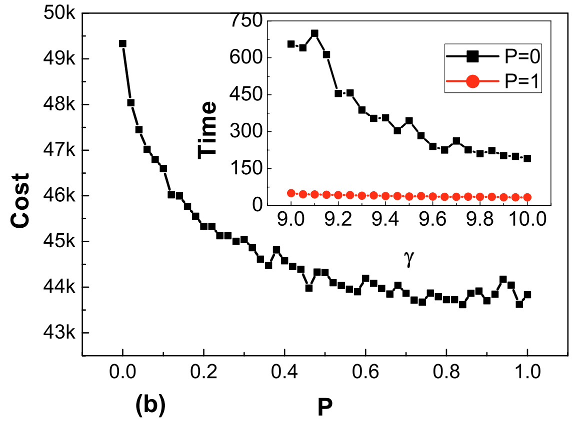

Fig. 5. Cost as a function of probability of randomization. The inset is the relaxation time as a function of coupling strength γ at $P = 0$ and $p = 1$.

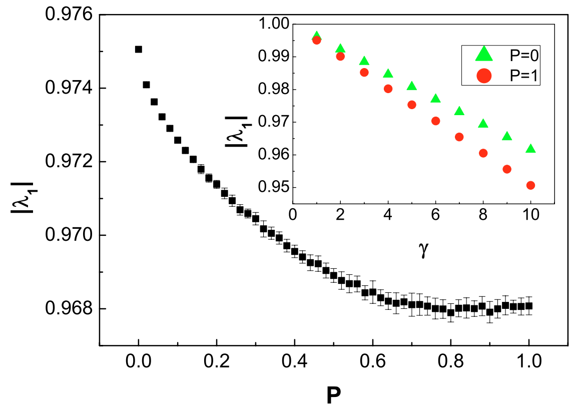

Fig. 6. Eigenvalue $|\lambda_1|$ as a function of probability of randomization with coupling strength $\gamma = 6.5$ ($N = 300$, $k = 30$). The inset shows eigenvalue $|\lambda_1|$ as a function of coupling strength γ. The error bars indicate the standard deviation of results.

to reach synchronization can only be estimated from the onset of synchronization, it is calculated within the range where complete synchronization could be achieved. The relaxation time for synchronization is shortened after randomizing link weights (as shown in the inset of Fig. 5).

For a sparse network (with small k), although the system cannot reach complete synchronization (as shown in Fig. 6), the decreasing $|\lambda_1|$ with randomization probability in regular networks is qualitatively similar to the above results. The decrease of average path length is smaller in sparse networks than in dense networks [18], so the change of $|\lambda_1|$ is small. Compared with the error, the decrease of $|\lambda_1|$ is much more significant. This implies that heterogeneous weight distribution can enhance the synchronizability regardless of the density of the regular network. Of course, with increasing density, the effect of weight randomization on synchronizability grows more obvious.

Besides studying linear stability of the synchronization states by investigating $|\lambda_1|$, we use the degree of synchronization ($D = \lim_{t\to\infty} \frac{1}{t} \sum_t \sum_i |x_i(t) - \overline{x}_t|$) to study how much and under what conditions they converge toward a coherent state. In the simulation process, the degree of synchronization (D) is estimated after eliminating a certain relaxation time, and then averaging over the following 1500 time steps. Clearly the complete synchronization appears if and only if the degree of synchronization (D) is zero within computer numerical precision, i.e. $D \sim 10^{-14}$. This procedure not only reveals the existence of stable solutions, but also gives a rough measure of its attracting basin. Figure 3b indicates that the degree of synchronization reaches zero at $P \simeq 0.4$, while the synchronization states become stable at $\gamma = 6.5$ in Figure 3a. This may imply that the system has only one synchronization attractor in the phase space. We could find a sharp transition to coherence, and a robustness to initial configurations, since for each overall coupling strength γ, all the final configurations have approximately the same degree of synchronization. In particular, when exceeding the threshold around $P \simeq 0.4$, all initial configurations converge toward a coherent state, indicating that in this parameter region the attracting basin of coherent states fills almost the entire phase space. This conclusion is confirmed in Figure 4. Figure 4b shows the curves of D as a function of coupling strength γ before and after redistribution of link weights. We can observe that D becomes synchronized at lower γ with a heterogeneous weight distribution when $P = 1$ from 50 different initial configurations. Similarly, comparing with Figure 4a, we can see that once the synchronization states are stable, the system becomes synchronized immediately. Further detailed studies about the basin of attractor in the phase space need to be performed.

From the above results, we believe that redistribution of weight could enhance the synchronizability of oscillators on the lattice. Meanwhile, we notice that heterogeneity is also increased by randomization of the weight. We therefore changed the technique for redistributing weight in order to check the role of link weights in the synchronization of oscillators on the lattice.

We distribute the link weights with a uniform distribution in $[10 - \theta, 10 + \theta]$, which means the weight average remains 10. The weight distribution becomes more heterogeneous as θ is increased. Figure 7 displays that with increasing coupling strength, $|\lambda_1|$ decreases more rapidly with a larger θ, while the degree of synchronization reaches zero at the smaller γ. This implies that the system becomes increasingly synchronizable with increasing heterogeneous weight distribution. This is consistent with the above results that the randomization of weight can enhance the synchronizability on regular networks. These results suggest that another efficient way could achieve better synchronizability on regular networks.

It should be noted that this study was performed only on regular networks. The influence of link weights on the synchronization with other topologies of network is unclear. However, these questions could be solved if we consider the weight and topology together. Despite its preliminary character, the study presented here can clearly

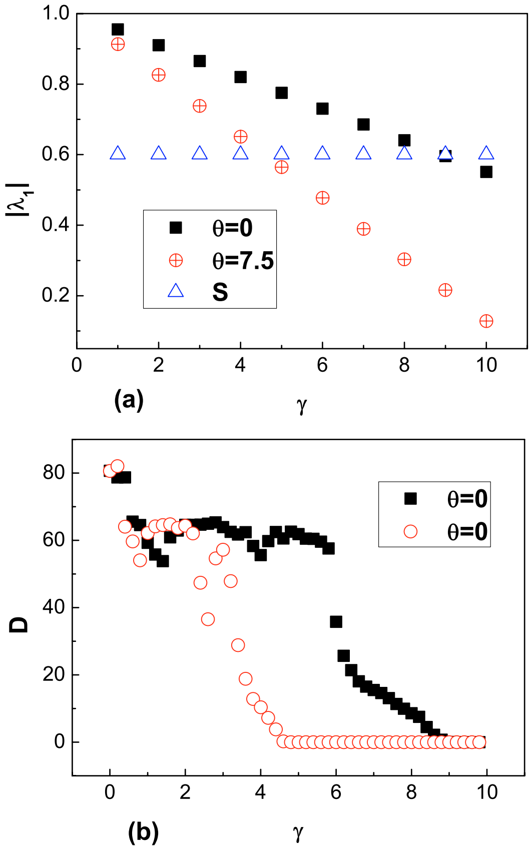

Fig. 7. (a) Eigenvalue $|\lambda_1|$ as a function of coupling strength with $\theta = 0$ and $\theta = 7.5$. (b) Degree of synchronization as a function of coupling strength with $\theta = 0$ and $\theta = 7.5$ respectively. It can also be observed that the threshold for stability is the same probability as the threshold for both $\theta = 0$ and $\theta = 7.5$.

indicate the role of heterogeneous weight distribution in the synchronization.

5 Conclusion

Link weight, as a measure of interaction strength, is believed to be an important variable in networks. It gives more information about networks besides topological properties dominated by links, and provides an additional tool to adjust network properties. For weighted networks, besides changing their topology by rewiring of links, redistribution of weight is an important way to improve the network function. In this paper, through a simple system of coupled Logistic maps, we have shown that randomization of link weights yields higher synchronizability on regular network at smaller cost. The Lyapunov exponent of the network is decreased with the randomization of link weights and the system reaches complete synchronization more easily. The reason for this may be that the average path length is shortened as a result of heterogeneity of link weights. The relations between oscillators are tightened so that the oscillators can synchronize easily. Results obtained by analysis of the coupling matrix were numerically analyzed in coupled Logistic maps. We studied the degree of synchronization of the system from random initial conditions. One interesting result is that the entire phase space is almost completely filled with the attracting basin of synchronous states. We also confirm that the heterogeneity of link weights can reinforce the synchronization ability for a uniform randomly redistribution of link weights. Thus our works has mainly focused on these effects in regular networks. The effect of redistribution of weight on other kinds of network should be investigated. The more interesting problem for further studies is to find the best distribution of link weights and link-weight correspondence for synchronization for a given choice of dynamics and network topology. This would be useful for network design and control of synchronization.

The work is partially supported by the 985 project and NSFC under the grant Nos. 70431002 and 70471080.

References

1. S. Boccaletti, J. Kurths, G. Osipov, D.L. Valladares, C.S. Zhou, Phys. Rep. **366**, 1 (2002); A. Pikovsky et al., Cambridge Nonlinear Science Series 12 (Cambridge University Press, UK, 2001)
2. S.H. Strogatz, Nature (London) **410**, 268 (2001)
3. R. Albert, A.L. Barabási, Rev. Mod. Phys. **74**, 47 (2002); S.N. Dorogovtsev, J.F. Mendes, Adv. Phys. **51**, 1079 (2002)
4. D.J. Watts, S.H. Strogatz, Nature (London) **393**, 440 (1998)
5. A.L. Barabási, R. Albert, Science **286**, 509 (1999)
6. L.F. Lago-Fernádez, R. Huerta, F. Corbacho, J.A. Sigüenza, Phys. Rev. Lett. **84**, 2758 (2000)
7. B. Barahona, L.M. Pecora, Phys. Rev. Lett. **89**, 054101 (2002)
8. T. Nishikawa, A.E. Motter, Y.-C. Lai, F.C. Hoppensteadt, Phys. Rev. Lett. **91**, 014101 (2003)
9. Sarika Jalan, R.E. Amritkar, Chin-Kun Hu, Phys. Rev. E. **72**, 016211/016212 (2005)
10. A.E. Motter, C.S. Zhou, J. Kurths, AIP Conference Proceedings **776**, 201 (2005)
11. M. Chavez, D.-U. Hwang, A. Amann, H.G.E. Hentschel, S. Boccaletti, Phys. Rev. Lett. **94**, 218701 (2005)
12. C.S. Zhou, J. Kurths, Phys. Rev. Lett. **96**, 164102 (2006)
13. J.W. Scannell et al., Cereb. Cortexv **9**, 277 (1999)
14. B.T. Grenfell, O.N. Bjornstad, J. Kappey, Nature (London) **414**, 716 (2001)
15. G. Korniss, M.A. Novotny, H. Guclu, Z. Toroczkai, P.A. Rikvold, Science **299**, 677 (2003)
16. A.E. Motter, C.S. Zhou, J. Kurths, Phys. Rev. E. **71**, 016116 (2005)
17. A.E. Motter, C.S. Zhou, J. Kurths, Europhys. Lett. **69**, 334 (2005)
18. M. Li, Y. Fan, D. Wang, N. Liu, D. Li, J. Wu, Z. Di, Phys. Lett. A **364**, 488 (2007)
19. P. Holme, S.M. Park, B.J. Kim, C.R. Edling, Physica A **373**, 821 (2007)
20. J.-P. Onnela, J. Saramaki, J. Kertesz, K. Kaski, Phys. Rev. E **71** 065103 (2005)