Effects of Weight on Structure and Dynamics in Complex Networks

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Abstract

Link weight is crucial in weighted complex networks. It provides additional dimension for describing and adjusting the properties of networks. The topological role of weight is studied by the effects of random redistribution of link weights based on regular network with initial homogeneous weight. The small world effect emerges due to the weight randomization. Its effects on the dynamical systems coupled by weighted networks are also investigated. Randomization of weight can increase the transition temperature in Ising model and enhance the ability of synchronization of chaotic systems dramatically.

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Network analysis is now widely used to describe the relationship and collective behavior in many fields[1, 2]. A binary network has a set of vertices and a set of edges which represent the relationships between any two vertices. Obviously, the edge only represents the presence or absence of interaction. It will be a strong limitation when such an approach is used to describe relations with different strength or having more levels. In fact, the interaction strength usually plays an important role in many real networks. For example, the number of passengers or flights between any two

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airports in airport networks[3, 4, 5], the closeness of any two scientists in scientific collaboration networks[6, 7, 8, 9], the reaction rates in metabolic networks[10] are all crucial in the characterization of the corresponding networks. So it is necessary to take the link weight into account and to study weighted networks. Recently more and more studies in complex networks focus on the weighted networks. The problems involve the definition of weight and other quantities which characterize the weighted networks[11, 12, 9], the empirical studies about its statistical properties[13, 14, 15, 8], evolving models[16, 17, 18, 19, 20], and transportation or other dynamics on weighted networks[21, 22, 23].

Weight, a quantity reflects the strength of interaction, gives more information about networks besides its topology dominated by links. Weight provides additional dimension to describe and to adjust network properties. However, how important is the weight, or what significant changes on network structures are induced when weight is changed? This question is related to the role of weight. It should be a fundamental problem in the study of weighted networks. But it has not been investigated deeply in previous studies.

For weighted networks, the redistribution of weight on links provide another way to adjust network structure besides the change of links between vertices. The role of weight could be studied both by its effects on network structures and its effects on dynamical processes taking place on networks. The effects of weight on network structures can be investigated on single vertex statistics, such as degree and clustering coefficient, and global properties such as distance, betweenness and especially community structure. The effects of weight on dynamics can be examined by the collective behaviors of dynamical systems, such as the synchronization of oscillators or chaos, the phase transition of spin system, the coherent oscillations of excitable systems, the spread of an infectious disease, the propagation of information, and so on. Except we consider the difference of above properties between unweighted and weighted networks, an efficient way to study the effects of weight is to consider the difference after disturbing the weight distribution. We have introduced the way to disturb the weight distribution and investigated its effects on network structures including single vertex statistics, distance[9], and community structure[24] in several real networks. The conclusion revealed that link weight has effects on network structures especially on the global properties. Similar to the method used by Watt and Strogatz in their discussion on small-world networks[25], here we present a more general method to investigate the effect of weight based on regular networks and an idealized construction. The results are interesting and valuable because real networks are usually weighted and the redistribution of weight on links provides another significant way to improve the structure and function of networks.

Effects of Weight on the Structure of Networks Usually there are two kinds of weight in weighted networks. One is dissimilar weight, such as the distance between two airports. Dissimilar weight has same meaning as the distance in binary networks. The bigger is the weight, the larger is the distance between two nodes. Another kind of weight is similar weight, such as the times of coauthoring of scientists in scientific collaboration networks and the flux in metabolic network. Similar weight has opposite meaning as the general distance in binary networks. The bigger weight means the smaller distance between two vertices. If the weight is defined in sense of similarity, the calculation of such similarity for example between two vertices connected by two edges (with w_1 and w_2 respectively) is $\tilde{w} = \frac{1}{\frac{1}{w_1}+\frac{1}{w_2}}$ [8]. In our following discussions, for simplicity and without losing any generalization, the weight on links is dissimilar weight with $w_{ij} \in [1, \infty)$ if not mentioned. Then the distance of a path can be easily get from the sum of weight.

For weighted networks, the generalization of Watts-Strogatz clustering coefficient is not entirely trivial. We must define a quantity to measure the strength of connections within i's neighborhood. B. J. Kim has argued that this quantity should fulfill four requirements and then provided a definition of weighted clustering coefficients[26]. In order to apply his definition to our weighted networks, we should first convert our dissimilar weight into similar weight $\tilde{w}_{ij} = \frac{1}{w_{ij}}$, then we have $\tilde{w}_{ij} \in [0, 1]$. Using the following equation

$$c_w(i) = \frac{\sum_{jk} \tilde{w}_{ij}\tilde{w}_{jk}\tilde{w}_{ki}}{\max_{ij} \tilde{w}_{ij} \sum_{jk} \tilde{w}_{ij}\tilde{w}_{ki}} \tag{1}$$

we can get the local clustering coefficient for every node i and then get the average clustering coefficient of the network.

In order to investigate the effects of weight on network structure, we consider the same methodology of constructing WS small-world networks. Instead of considering the link random rewiring procedure, we study the effects of random redistribution of weight on links for weighted regular network. Starting from a ring lattice with N vertices and k edges per vertex, each edge has a same weight w=5 in the initial state. Firstly, we divide the weight into a smaller unit Δw ($\Delta w = 1$). Secondly, we extract randomly each unit with probability p. Lastly, we equiprobably lay back each unit to

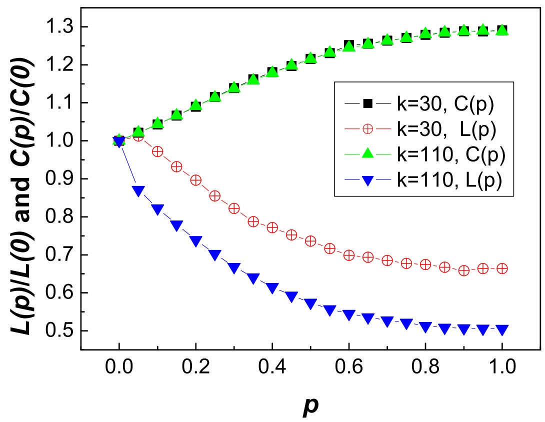

Figure 1: Characteristic path length $L(p)/L(0)$ and clustering coefficient $C(p)/C(0)$ for the family of randomly weight redistributed networks ($N = 200$). k is the edges per vertex. All the results (including Fig.2) are average over 20 random realizations of the redistribution process.

all links. Without changing of links among vertices, this construction allows us to have a regular network with uniform weight distribution (p=0, δ distribution) and random weight distribution (p=1, Gaussian distribution). And through the investigation of the intermediate region $0 < p < 1$, we can know the effects of weight redistribution.

The same as WS small-world network, we use average path length $L(p)$ and clustering coefficient $C(p)$ to quantify the structural properties of the network. Fig. 1 reveals that with the random redistribution of link weight, the average path length decreases obviously, while the average clustering coefficient increases. So it gives the similar phenomenon as small-world effect, but here it is caused by the random redistribution of weight instead of rewiring of links.

It seems that above effects of weight redistribution are more significant in dense networks than in sparse networks. For a given randomization probability p, Fig. 2 (a) gives the results of $L(p)$ as a function of degree k. With the increase of the denseness of network, the decrease of $L(p)$ becomes more and more obvious and $L(p)$ reaches an extremum to some extent. Then the effect becomes less with the following increase of k.

It is interesting to notice that above effects are not related with the scale of networks. We try this procedure for other networks with different scales

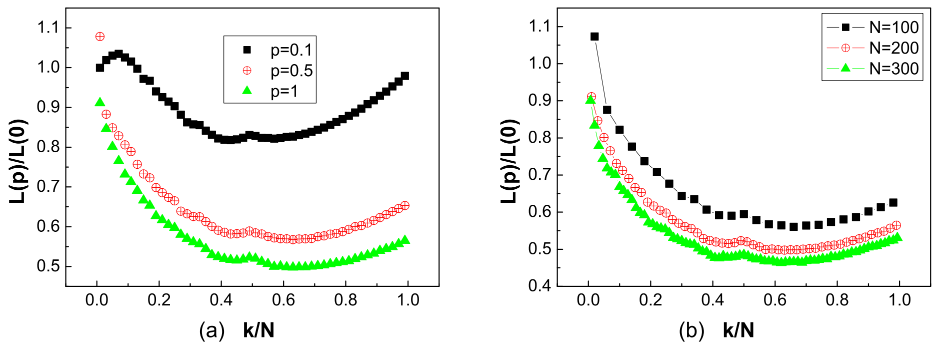

Figure 2: (a) With a given random redistribution probability p, $L(p)/L(0)$ as a function of network denseness described by k/N ($N = 200$). (b)For networks with different scale, the curves are almost the same ($p = 1$).

(number of vertices N). If we re-scale the system with the quantity k/N (which describes the denseness of networks), we could get similar curves for $L(p)/L(0)$. This demonstrates the robustness of the results (Fig 2(b)).

The idealized construction above gives the small-world phenomenon due to the random redistribution of weight in networks. In order to know the detailed change of network structure, we examine the distribution of vertex and link betweenness after redistributing weight. As shown in Fig. 3, from the initial homogeneous case, the distribution of link betweenness becomes power-law and the distribution of vertex betweenness turns into Γ distribution. The result shows that some hubs emerge as a result of the weight randomization. Hence the predicted changes in the structure indicate that we have changed the global structural features due to the only random redistribution of weight without changing connections.

Effects of weight randomization on Dynamics For investigating the functional significance of weight redistribution, we impose this method on spin systems and synchronization of chaotic systems on networks.

A. Spin Systems The phase transition of Ising model on many kinds of networks have been studied deeply. In small-world network, the transition temperature T_c of Ising model increases with the rewiring probability p [28, 29]. In most previous studies of spin model on networks, spin interactions have been assumed to be uniform. But in reality, interactions are different with each other. In reference [27], the geometrical distance is considered as the interaction strength between any two nodes i and j. In this letter, we

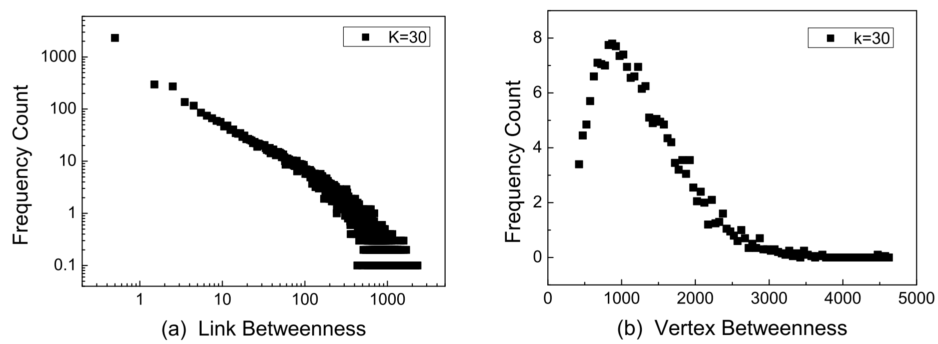

Figure 3: Distribution of link (a) and node (b) betweenness after the full random redistribution of weight$(p = 1)$.

consider both nearest-neighbor and next nearest-neighbor interactions[30] on weighted regular networks. The Hamiltonian for Ising model on network is given by

$$H = -\frac{1}{2}\sum_{i \neq j} J_{ij}\sigma_i\sigma_j, \tag{2}$$

where $\sigma_i(=\pm 1)$ is the Ising spin on node i. Given the dissimilar weight w_{ij} between any two nodes i and j connected directly, the interaction J_{ij} reads $1/w_{ij}$ if i and j are connected directly, reads $1/min_s(w_{is} + w_{sj})$ if i and j are next nearest-neighbor and reads 0 for other conditions. The transition process is described by the magnetization $M = \frac{1}{N}\sum_{i=1}^{N}\sigma_i$. From a given temperature T and random initial spin state, we perform annealing algorithm to describe the evolution of the system. In the process, if $\Delta H = H_{new} - H_{old} > 0$, transition probability from low energy state to high energy state is $exp(\frac{-\Delta H}{T})$.

Starting from a ring lattice with $N = 500$ vertices and $k = 200$ edges per vertex, each link has same weight w=1 in the initial network. Then we redistribute each $\Delta w = 0.1$ randomly with probability p. We find that the phase transition courses vary with different redistribution probability p (Fig. 4). It shows that the randomization of weight has induced the increase of critical temperature T_c. The redistribution of weight has similar effects as the rewiring of links on phase transition.

B. Synchronization of Chaotic System Since their introduction in 1989 [31], coupled maps have been studied as a paradigmatic example in the study of the emergent behavior of complex systems as diverse as eco-

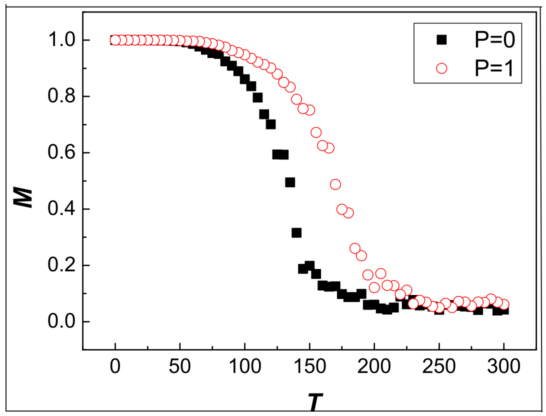

Figure 4: Magnetization M evolves with temperature T for homogeneous ($p = 0$) and fully randomization ($p = 1$) of weight distribution. All the results are the average over 20 runs from different random initial configurations of spins.

logical networks, the immune system, or neural and cellular networks. In recent years, the synchronization of chaotic system on complex networks has drawn much attention. In [32], chaotic system on randomly coupled maps is investigated. It is found that the synchronization properties of the system are strongly dependent on the particular architecture. Graphs with the same number connectivity might have very different collective behavior. The previous works focused mainly on the effect of topology of network on the synchronization. Here we investigate regular networks of chaotic maps connected symmetrically and mainly focus the influence of redistributing weight on synchronization. We take the following coupled map

$$x_i(t+1) = (1-\varepsilon_i)f(x_i(t)) + \frac{\lambda}{m}\sum_{i\neq j} J_{ij} f(x_j(t)), \tag{3}$$

where $x_i(t)$ is the state variable and t denotes the discrete time. $f(x)$ prescribes the local dynamics, and is chosen as the logistic map $f(x_i) = \alpha x_i(1-x_i)$ with $\alpha = 3.9$. $\varepsilon_i = \frac{\lambda}{m}\sum_{i\neq j} J_{ij}$ gives the long-range coupling strength, where the sum is taken over all the m coupling nodes with i. We consider both nearest-neighbor and next nearest-neighbor couplings and take the same J_{ij} as in Ising system and parameters are $N = 300$, $k = 120$, $w = 1$ and $\Delta w = 0.1$. We make use of Degree of Synchronization

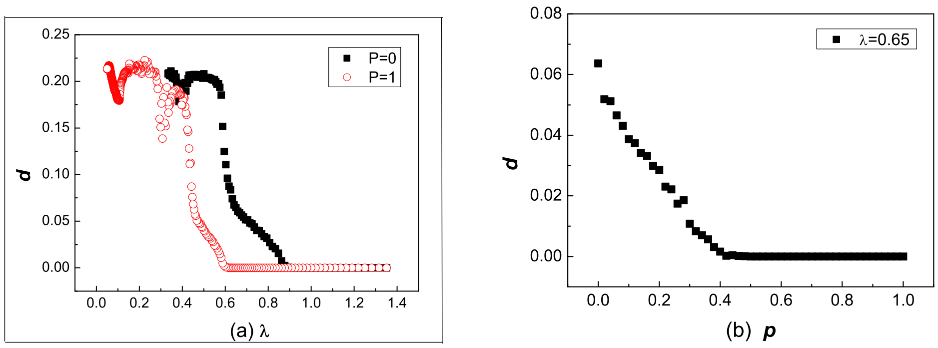

Figure 5: (a) Final d as the function of λ for homogeneous ($p = 0$) and fully randomization ($p = 1$) of weight distribution. (b) The final d as the function of probability p with $\lambda = 0.65$.

$d = \frac{1}{N}\sum_i^N |x_i - E(x_i)|$ to discriminant whether to reach synchronization or not. In the simulation, we neglect 500 steps as the transitional process and d is the average of following 1500 steps. All the results are the average over 50 runs from different initial conditions.

We first compare the final Degree of Synchronization d of homogeneous ($p = 0$) and fully randomization ($p = 1$) of weight distribution for given average weight. The results (Fig.5 (a)) show that the redistribution of weight is helpful to the synchronization of system. Fig.5 (b) gives the final d as the function of probability p. The randomization of weight enhances the ability of synchronization.

In this letter, we emphasize the effects of weight on the structural properties and function of networks. We explore a simple model of networks with regular connections and homogeneous weight. Instead of rewiring links, we introduce a method of randomly redistributing weight of links. Its effects are investigated both on structural properties and dynamical systems. With the random redistribution of weight, we observed the similar properties as in WS small-world networks. That is the average path length declines while the clustering coefficient is increased. We have also found that the random redistribution of weight can lead to the increase of critical temperature in Ising model and enhance the ability of synchronization of coupled chaotic systems.

Different from the most previous research on complex networks, which focus on the topological structure and its influences to the dynamical process, our investigation here illuminates the effects of weight. From the previous study on dynamical systems on networks with homogeneous coupling, it is already known that the variation of link weight will affect the global function of networks. But our results demonstrate that the change of weight distribution can also cause some significant effects on the subtle structural features and the functions of the given networks. These results reveal that network topology coupled by weight distribution should be essential to understand the structural properties and function of weighted networks in real world.

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